

6D superconformal theory as the Theory of Everything.

A.V. SMILGA

*SUBATECH, Université de Nantes, 4 rue Alfred Kastler, BP 20722, Nantes 44307,
France **

e-mail: `smilga@subatech.in2p3.fr`

Abstract

We argue that the fundamental Theory of Everything is a conventional field theory defined in the *flat* multidimensional bulk. Our Universe should be obtained as a 3-brane classical solution in this theory. The renormalizability of the fundamental theory implies that it involves higher derivatives (HD). It should be supersymmetric (otherwise one cannot get rid of the huge induced cosmological term) and probably conformal (otherwise one can hardly cope with the problem of ghosts). We present arguments that in conformal HD theories the ghosts (which are inherent for HD theories) might be not so malignant. In particular, we present a nontrivial QM HD model where ghosts are *absent* and the spectrum has a well defined ground state.

The requirement of superconformal invariance restricts the dimension of the bulk to be $D \leq 6$. We suggest that the TOE lives in six dimensions and enjoys the maximum $\mathcal{N} = (2, 0)$ superconformal symmetry. Unfortunately, no renormalizable field theory with this symmetry is presently known. We construct and discuss an $\mathcal{N} = (1, 0)$ 6D supersymmetric gauge theory with four derivatives in the action. This theory involves a dimensionless coupling constant and is renormalizable. At the tree level, the theory enjoys conformal symmetry, but the latter is broken by quantum anomaly. The sign of the β function corresponds to the Landau zero situation.

1 Motivation

Arguably, the most burning unresolved problem of modern theoretical physics is the absence of a satisfactory quantum theory of gravity. The *main* obstacle here is the geometric nature of gravity. Time is intertwined there with spatial coordinates and the notion of universal flat time is absent. As a result, in contrast to conventional field theory, one cannot write the (functional) Schrödinger equation, define the Hilbert space with unitary evolution operator, etc.

As a matter of fact, Einstein gravity (and any other theory where the metric is considered as a fundamental dynamic variable) has problems also at the classical level. The equations of motion cannot be *always* formulated as Cauchy problem. This leads to breaking of causality for some exotic configurations like Gödel universes or wormholes [1].

*On leave of absence from ITEP, Moscow, Russia.

Even though these configurations are not realized in our world at the macroscopic level, their existence presents conceptual difficulties.

The modern paradigm is that the fundamental Theory of Everything is a form of string theory. If this is true, gravity has the status of effective theory and one is not allowed to blame it for inconsistencies. But string theory also does not provide a satisfactory answer to all these troubling questions. Actually, they cannot even be posed there: we understand more or less well what string theory is only at the perturbative level (and even there we are not sure yet whether technical difficulties preventing one now to perform calculation of string amplitudes beyond two loops can be efficiently resolved), while its non-perturbative formulation is simply absent.

This has led us to suggest [2] that the TOE is a *field theory* living in *flat* higher-dimensional space. This higher-dimensional theory should involve 3-brane classical solutions, which might be associated with our Universe in the spirit of [3]. The gravity is induced there as an effective theory living on the brane. One can imagine a thin soap bubble. Its effective hamiltonian is

$$H^{\text{eff}} = \sigma \int \sqrt{g} d^2x , \quad (1.1)$$

where σ is the surface tension. The hamiltonian (1.1) is geometric, but the fundamental theory of soap is not: it is formulated in flat 3D space and does not know anything about the metric, etc. Of course, the analogy is not exact because the effective hamiltonian (1.1) does not have an Einstein form but looks rather as a cosmological term. The Einstein term and also the terms involving higher powers of curvature appear as corrections, however. In the observable world, the cosmological term is either zero or very small and one should think of a mechanism to get rid of it. One could succeed in that (if any) only if the fundamental theory is supersymmetric. Indeed, only supersymmetry can provide for the exact cancellation of quantum corrections to the energy density of the brane solution.

If we want the fundamental higher-dimensional theory to be renormalizable, the canonical dimension of the lagrangian should be greater than 4, i.e. it should involve higher derivatives. HD theories are known to have a problem of ghosts, which in many cases break unitarity and/or causality of the theory. [4]¹ However, a model study performed in Refs. [2, 5] indicates that in some cases, namely, when the theory enjoys *exact* conformal invariance, the ghosts are not so malignant, a well defined ground state (the vacuum) might exist and the theory might enjoy a unitary S-matrix.

We conclude that the TOE should be superconformal theory. This restricts the number of dimensions D in the flat space-time where the theory is formulated by $D \leq 6$. Indeed, all superconformal algebras involving the super-Poincare algebra as a subalgebra are classified [6]. Their highest possible dimension is six, which allows for the minimal conformal superalgebra (1,0) and the extended chiral conformal superalgebra (2,0).

Our hypothesis is that the TOE lives in six dimensions and enjoys the highest possible supersymmetry (2,0).

¹Physically, a ghost-ridden theory is simply a theory where the spectrum has no bottom and one cannot define what vacuum is.

Unfortunately, no field theory with this symmetry group is actually known now. The corresponding lagrangian is not constructed, and only indirect results concerning scaling behavior of certain operators have been obtained so far [7]. In [8], we derived (using the formalism of harmonic superspace (HSS) [9]) the lagrangian for the $6D$ gauge theory with unextended $(1,0)$ superconformal symmetry. This theory is conformal at the classical level and renormalizable. However, it is not finite: the β function does not vanish there and conformal symmetry is broken at the quantum level by anomaly. In other words, this theory cannot be regarded as a viable candidate for the TOE. Its study represents, however, a necessary step before the problem of constructing and studying the $(2,0)$ theory could be tackled.

In the next section, we explain in more details *what* are the ghosts, why (if not dealt with) they make the theory sick, and also present a special QM HD model where the ghosts *are* tamed. In sect. 3 we derive the lagrangian of our superconformal $6D$ theory and calculate its beta function. The last section is devoted, as usual, to conclusions and speculations.

2 Ghost-free QM higher derivative model.

To understand the nature of ghosts, one does not need to study field theories. It is clearly seen in toy models with finite number of degrees of freedom. Consider e.g. the lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{\Omega^4}{2}q^2. \quad (2.1)$$

It is straightforward to see that four independent solutions to the corresponding classical equations of motion are $q_{1,2}(t) = e^{\pm i\Omega t}$, $q_3(t) = e^{-\Omega t}$ and $q_4(t) = e^{\Omega t}$. The exponentially rising solution $q_4(t)$ displays instability of the classical vacuum $q = 0$. The quantum hamiltonian of such a system is not hermitian and the evolution operator $e^{-i\hat{H}t}$ is not unitary.

This vacuum instability is characteristic for all *massive* HD field theories — the dispersive equation has complex solutions in this case for small enough momenta. But for intrinsically massless (conformal) field theories the situation is different. Consider the lagrangian

$$L = \frac{1}{2}(\ddot{q} + \Omega^2 q)^2 - \frac{\alpha}{4}q^4 - \frac{\beta}{2}q^2\dot{q}^2. \quad (2.2)$$

Its quadratic part can be obtained from the HD field theory lagrangian $\mathcal{L} = (1/2)\phi\Box^2\phi$ involving massless scalar field, when restricting it on the modes with a definite momentum \vec{k} ($\Omega^2 = \vec{k}^2$). If neglecting the nonlinear terms in (2.2), the solutions of the classical equations of motion $q(t) \sim e^{\pm i\Omega t}$ and $q(t) \sim te^{\pm i\Omega t}$ do not involve exponential instability, but include only comparatively “benign” oscillatory solutions with linearly rising amplitude.

We showed in [2] that, when nonlinear terms in Eq.(2.2) are included, an island of stability in the neighbourhood of the classical vacuum ²

$$q = \dot{q} = \ddot{q} = q^{(3)} = 0 \quad (2.3)$$

²Usually, the term classical vacuum is reserved for the point in the configuration (or phase) space with

exists in a certain range of the parameters α, β . In other words, when initial conditions are chosen at the vicinity of this point, the classical trajectories $q(t)$ do not grow, but display a decent oscillatory behaviour. This island is surrounded by the sea of instability, however. For generic initial conditions, the trajectories become singular: $q(t)$ and its derivatives reach infinity in a finite time.

Such a singular behaviour of classical trajectories often means trouble also in the quantum case. A well-known example when it does is the problem of 3D motion in the potential

$$V(r) = -\frac{\gamma}{r^2}. \quad (2.4)$$

The classical trajectories where the particle falls to the centre (reaches the singularity $r = 0$ in a finite time) are abundant. This occurs when $l > \sqrt{2m\gamma}$, where l is the classical angular momentum. And it is also well known that, if $m\gamma > 1/4$, the quantum problem is not very well defined: the eigenstates with arbitrary negative energies exist and the hamiltonian does not have a ground state.

The bottomlessness of the quantum hamiltonian is not, however, a necessary corollary of the fact that the classical problem involves singular trajectories. In the problem (2.4), the latter are present for all positive γ , but the quantum ground state disappears only when γ exceeds the boundary value $1/(4m)$.

Our main observation here is that the system (2.2) exhibits a similar behaviour. If both α and β are nonnegative (and at least one of them is nonzero), the quantum hamiltonian has a bottom and the quantum problem is perfectly well defined even though some classical trajectories are singular.

2.1 Free theory

Before analyzing the full nonlinear system (2.2), let us study the dynamics of the truncated system with the lagrangian $L = (\ddot{q} + \Omega^2 q)^2/2$. As was observed in [10], this system displays a singular behavior. It is instructive to consider first the lagrangian

$$L = \frac{1}{2} [\ddot{q}^2 - (\Omega_1^2 + \Omega_2^2)\dot{q}^2 + \Omega_1^2\Omega_2^2 q^2] \quad (2.5)$$

and look what happens in the limit $\Omega_1 \rightarrow \Omega_2$. When $\Omega_1 > \Omega_2$, the spectrum of the theory (2.5) is

$$E_{nm} = \left(n + \frac{1}{2}\right) \Omega_1 - \left(m + \frac{1}{2}\right) \Omega_2 \quad (2.6)$$

with nonnegative integer n, m . On the other hand, when $\Omega_1 = \Omega_2 = \Omega$, the spectrum is

$$E_n = n\Omega \quad (2.7)$$

minimal energy. For HD theories and in particular for the theory (2.2) the classical energy functional is not bounded from below and by “classical vacuum” we simply mean a stationary solution to the classical equations of motion.

with generic integer n . In both cases, the quantum hamiltonian has no ground state, but in the limit of equal frequencies the number of degrees of freedom is apparently reduced in a remarkable way: instead of two quantum numbers n, m (the presence of two quantum numbers is natural — the phase space of the system (2.5) is 4-dimensional having two pairs $(p_{1,2}, q_{1,2})$ of canonic variables), we are left with only one quantum number n .

This deficiency of the number of eigenstates compared to natural expectations would not surprise a mathematician. A generic 2×2 matrix has two different eigenvectors. But the Jordan cell $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only *one* eigenvector $\propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The statement is therefore that in the limit $\Omega_1 = \Omega_2$ our hamiltonian represents a kind of generalized Jordan cell.

Actually, the “lost” degrees of freedom reinstall themselves when taking into account nontrivial *time dynamics* of the degenerate system (2.5) with $\Omega_1 = \Omega_2$. The situation is rather similar to what has been unravelled back in the sixties when studying degenerate systems displaying “nonexponential decay” behavior (see e.g. [11]). We will not discuss here nonstationary problem and concentrate on the *spectrum* of this system.

To begin with, let us construct the canonical hamiltonian corresponding to the lagrangian (2.5). This can be done using the general Ostrogradsky formalism [12]³. For a lagrangian like (2.5) involving q, \dot{q} , and \ddot{q} , it consists in introducing the new variable $x = \dot{q}$ and writing the hamiltonian $H(q, x; p_q, p_x)$ in such a way that the classical Hamilton equations of motion would coincide after excluding the variables x, p_x, p_q with the equations of motion

$$q^{(4)} + (\Omega_1^2 + \Omega_2^2)\ddot{q} + \Omega_1^2\Omega_2^2q = 0 \quad (2.8)$$

derived from the lagrangian (2.5). This hamiltonian has the following form

$$H = p_q x + \frac{p_x^2}{2} + \frac{(\Omega_1^2 + \Omega_2^2)x^2}{2} - \frac{\Omega_1^2\Omega_2^2q^2}{2} . \quad (2.9)$$

For example, the equation $\partial H / \partial p_q = \dot{q}$ gives the constraint $x = \dot{q}$, etc.

When $\Omega_1 \neq \Omega_2$, the quadratic hamiltonian (2.9) can be diagonalized by a certain canonical transformation $x, q, p_x, p_q \rightarrow a_{1,2}, a_{1,2}^*$ [5, 10]. We obtain

$$H = \Omega_1 a_1^* a_1 - \Omega_2 a_2^* a_2 . \quad (2.10)$$

The classical dynamics of this hamiltonian is simply $a_1 \propto e^{-i\Omega_1 t}$, $a_2 \propto e^{i\Omega_2 t}$. Its quantization gives the spectrum (2.6). The negative sign of the second term in (2.10) implies the negative sign of the corresponding kinetic term, which is usually interpreted as the presence of the ghost states (the states with negative norm) in the spectrum. We prefer to keep the norm positive definite, with the creation and annihilation operators $a_{1,2}, a_{1,2}^\dagger$ (that correspond to the classical variables $a_{1,2}, a_{1,2}^*$) satisfying the usual commutation relations $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1$. However, irrespectively of whether the metric is kept positive definite or not and the world “ghost” is used or not, the spectrum (2.6) does not have a ground state and, though the spectral problem for the free hamiltonian (2.10) is

³See e.g. [13] for its detailed pedagogical description.

perfectly well defined, the absence of the ground state leads to a trouble, the falling to the centre phenomenon when switching on the interactions.⁴

We are interested, however, not in the system (2.5) as such, but rather in this system in the limit $\Omega_1 = \Omega_2$. As was mentioned, this limit is singular. The best way to see what happens is to write down the explicit expressions for the wave functions of the states (2.6) and explore their behaviour in the equal frequency limit. This can be done by substituting the operators $-i\partial/\partial x$, $-i\partial/\partial q$ for p_x and p_q in Eq. (2.9) and searching for the solutions of the Schrödinger equation in the form

$$\Psi(q, x) = e^{-i\Omega_1\Omega_2qx} \exp \left\{ -\frac{\Delta}{2} (x^2 + \Omega_1\Omega_2q^2) \right\} \phi(q, x) , \quad (2.11)$$

where $\Delta = \Omega_1 - \Omega_2$. Then the operator acting on $\phi(q, x)$ is

$$\tilde{H} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + (\Delta x + i\Omega_1\Omega_2q) \frac{\partial}{\partial x} - ix \frac{\partial}{\partial q} + \frac{\Delta}{2} . \quad (2.12)$$

It is convenient to introduce

$$z = \Omega_1 q + ix , \quad u = \Omega_2 q - ix , \quad (2.13)$$

after which the operator (2.12) acquires the form

$$\tilde{H}(z, u) = \frac{1}{2} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial u} \right)^2 + \Omega_1 u \frac{\partial}{\partial u} - \Omega_2 z \frac{\partial}{\partial z} + \frac{\Delta}{2} . \quad (2.14)$$

The holomorphicity of $\tilde{H}(z, u)$ means that its eigenstates are holomorphic functions $\phi(z, u)$. An obvious eigenfunction with the eigenvalue $\Delta/2$ is $\phi(z, u) = \text{const}$. Further, if assuming ϕ to be the function of only one holomorphic variable u or z , the equation $\tilde{H}\phi = E\phi$ acquires the same form as for the equation for the preexponential factor in the standard oscillator problem. Its solutions are Hermit polynomials,

$$\begin{aligned} \phi_n(u) &= H_n(i\sqrt{\Omega_1}u) \equiv H_n^+, & E_n &= \frac{\Delta}{2} + n\Omega_1 , \\ \phi_m(z) &= H_m(\sqrt{\Omega_2}z) \equiv H_m^-, & E_m &= \frac{\Delta}{2} - m\Omega_2 . \end{aligned} \quad (2.15)$$

The solutions (2.15) correspond to excitations of only one of the oscillators while another one is in its ground state. For sure, there are also the states where both oscillators are excited. One can be directly convinced that the functions

$$\begin{aligned} \phi_{nm}(u, z) &= \sum_{k=0}^m \left(\frac{i\Delta}{4\sqrt{\Omega_1\Omega_2}} \right)^k \frac{(n-m+k+1)!}{(m-k)!k!} H_{n-m+k}^+ H_k^-, & m \leq n , \\ \phi_{nm}(u, z) &= \sum_{k=0}^n \left(\frac{i\Delta}{4\sqrt{\Omega_1\Omega_2}} \right)^k \frac{(m-n+k+1)!}{(n-k)!k!} H_k^+ H_{m-n+k}^-, & m > n \end{aligned} \quad (2.16)$$

⁴A characteristic feature of this phenomenon is that some classical trajectories reach singularity in a finite time while the quantum spectrum involves a *continuum* of states with arbitrary low energies [14]. In our case, the “centre” is not a particular point in the configuration (phase) space but rather its boundary at infinity, but the physics is basically the same.

are the eigenfunctions of the operator (2.14) with the eigenvalues (2.6). Multiplying the polynomials (2.16) by the exponential factors as distated by Eq.(2.11), we arrive at the normalizable wave functions of the hamiltonian (2.9).

We are ready now to see what happens in the limit $\Omega_1 \rightarrow \Omega_2$ ($\Delta \rightarrow 0$). Two important observations are in order.

- The second exponential factor in (2.11) disappears and the wave functions cease to be normalizable.
- We see that in the limit $\Delta \rightarrow 0$, only the first terms survive in the sums (2.16) and we obtain

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \phi_{nm} &\sim H_{n-m}^+ \quad , \quad m \leq n \\ \lim_{\Delta \rightarrow 0} \phi_{nm} &\sim H_{m-n}^- \quad , \quad m > n . \end{aligned} \quad (2.17)$$

In other words, the wave functions depend only on the difference $n - m$, which is the only relevant quantum number in the limit $\Omega_1 = \Omega_2$.

As this phenomenon is rather unusual and very important for us, let us spend few more words to clarify it. Suppose Ω_1 is very close to Ω_2 , but still not equal. Then the spectrum includes the sets of nearly degenerate states. For example, the states $\Psi_{00}, \Psi_{11}, \Psi_{22}$, etc have the energies $\Delta/2, 3\Delta/2, 5\Delta/2$, etc, which are very close. In the limit $\Delta \rightarrow 0$, the energy of all these states coincides, but rather than having an infinite number of degenerate states, we have only one state: the wave functions $\Psi_{00}, \Psi_{11}, \Psi_{22}$, etc simply *coincide* in this limit by the same token as the eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ \Delta & 1 \end{pmatrix}$ coincide in the limit $\Delta \rightarrow 0$.

2.2 Interacting theory.

When $\Omega_1 = \Omega_2$, $u = \bar{z}$ and the operator (2.14) acquires the form

$$\tilde{H}(z, \bar{z}) = \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z} \right)^2 + \Omega \left(\bar{z} \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial z} \right) . \quad (2.18)$$

Its spectrum is bottomless. Let us deform (2.18) by adding there the quartic term $\alpha z^2 \bar{z}^2$ with positive α . Note first of all that it cannot be treated as a perturbation, however small α is: the wave functions are not normalizable and the matrix elements of $\alpha z^2 \bar{z}^2$ diverge. But one can use the variational approach. Let us take the Ansatz

$$|\text{var}\rangle = z^n e^{-Az\bar{z}} , \quad (2.19)$$

where A, n are the variational parameters. The matrix element of the unperturbed quadratic hamiltonian (2.18) over the state (2.19) is

$$\langle \text{var} | \tilde{H} | \text{var} \rangle = \frac{A(n+1)}{2} - \Omega n . \quad (2.20)$$

Obviously, by choosing n large enough and A small enough, one can make it as close to $-\infty$ as one wishes. The bottom is absent and one cannot reach it. For the deformed hamiltonian, the situation is different, however. We have

$$E^{\text{var}}(n, A) = \langle \text{var} | \tilde{H} + \alpha z^2 \bar{z}^2 | \text{var} \rangle = \frac{A(n+1)}{2} - \Omega n + \frac{\alpha(n+1)(n+2)}{4A^2} . \quad (2.21)$$

This function has a global minimum. It is reached when

$$A - \Omega - \frac{\alpha}{4A^2} = 0 \quad (2.22)$$

and $n = A^3/\alpha - 2$.

For small $\alpha \ll \Omega^3$,

$$A \approx \Omega, \quad n \approx \frac{\Omega^3}{\alpha}, \quad \text{and} \quad E^{\text{var}} \approx -\frac{\Omega^4}{4\alpha} . \quad (2.23)$$

The smaller is α , the lower is the variational estimate for the ground state energy and the ground state energy itself. In the limit $\alpha \rightarrow 0$, the spectrum becomes bottomless. But for a finite α , the bottom exists. Note that in the interacting system, the spectrum is completely rearranged compared to the HD oscillator studied above and there is no reason to expect the peculiar Jordan-like degeneracy anymore. The eigenstates are conventional normalized functions and the solution of the time-dependent Schrödinger equation has the standard form.

Bearing in mind that $z = \Omega q + ix = \Omega q + i\dot{q}$, the deformation $\alpha z^2 \bar{z}^2$ amounts to a particular combination of the terms $\sim q^4$, $\sim q^2 \dot{q}^2$, and $\sim \dot{q}^4$ in the hamiltonian. For the theory (2.2) with generic α, β , the algebra is somewhat more complicated, but the conclusion is the same: in the case when the form $\alpha q^4/4 + \beta q^2 x^2/2$ is positive definite, the system has a ground state.

The requirement of positive definiteness of the deformation is necessary. In the opposite case, choosing the Ansatz

$$| \text{var} \rangle \sim (\Omega q + ix)^n \exp\{-Aq^2 - Bx^2\}$$

and playing with A, B , one can always make the matrix element $\langle \text{var} | \text{deformation} | \text{var} \rangle$ negative, which would add to the negative contribution $-\Omega n$ in the variational energy, rather than compensate it. The bottom is absent in this case.

3 Superconformal $6D$ theory

We start with reminding some basic facts of life for spinors in $SO(5, 1)$ (or rather $Spin(5, 1)$) . There are two different complex 4-component spinor representations, the $(1, 0)$ spinors ψ^a and the $(0, 1)$ spinors ξ_a . In the familiar $Spin(3, 1)$ case, there are also two different spinor representations, which are transformed to each other under complex conjugation (on the other hand, complex conjugation leaves an Euclidean $4D$ spinor in the same representation). An essential distinguishing feature of $Spin(5, 1)$ is that complex conjugation

does *not* change the type of spinor representation there (while it does for Euclidean $6D$ spinors, $Spin(6) \equiv SU(4)$).

Indeed, one can show that the spinor

$$\bar{\psi}^a = -C_a^a \psi^{\dot{a}}, \quad (3.1)$$

is transformed in the same way as ψ^a . We defined $\psi^{\dot{a}} = (\psi^a)^*$ and introduced a symplectic charge-conjugation matrix C satisfying

$$C_a^a C_b^{\dot{a}} = -\delta_b^a. \quad (3.2)$$

The operation $\bar{}$ is the covariant conjugation. A somewhat unusual property $\overline{\bar{\psi}^a} = -\psi^a$ holds.

Bearing in mind, however, that ψ^a and $\bar{\psi}^a$ belong to the same representation, it is very convenient [15] to treat them on equal footing and introduce $\psi_{i=1,2}^a = (\psi^a, \bar{\psi}^a)$. The relation

$$\bar{\psi}_i^a = \psi^{ai} = \epsilon^{ij} \psi_{aj} \quad (3.3)$$

holds.

We choose the antisymmetric representation of the $6D$ Weyl matrices

$$(\gamma^M)_{ab} = -(\gamma^M)_{ba} \quad \tilde{\gamma}_M^{ab} = \frac{1}{2} \varepsilon^{abcd} (\gamma_M)_{cd} \quad (3.4)$$

where $M = 0, 1, \dots, 5$ and ε^{abcd} is the totally antisymmetric symbol. The basic relations for these Weyl matrices are

$$(\gamma_M)_{ac} (\tilde{\gamma}_N)^{cb} + (\gamma_N)_{ac} (\tilde{\gamma}_M)^{cb} = -2\delta_a^b \eta_{MN}, \quad (3.5)$$

$$\varepsilon_{abcd} = \frac{1}{2} (\gamma^M)_{ab} (\gamma_M)_{cd}, \quad (3.6)$$

where η_{MN} is the metric of the $6D$ Minkowski space ($\eta_{00} = -\eta_{11} = \dots = -\eta_{55} = 1$) and $\gamma_M = \eta_{MN} \gamma^N$.

The generators of the $(1,0)$ spinor representation are $S^{MN} = -\frac{1}{2} \sigma^{MN}$, where

$$(\sigma^{MN})_a^b = \frac{1}{2} (\tilde{\gamma}^M \gamma^N - \tilde{\gamma}^N \gamma^M)_a^b, \quad \overline{\sigma^{MN}} = \sigma^{MN}. \quad (3.7)$$

Supersymmetric field theories are most naturally formulated in the framework of superspace approach. The $6D$ superspace is more complicated than the 4-dimensional one. A simple-minded $6D$ superspace involves, besides 6 bosonic coordinates, 8 fermionic coordinates θ_i^a . However, one can effectively reduce the number of fermionic coordinates using the *harmonic superspace* approach and working with *Grassmann analytic* superfields [9]. We are not able to dwell on this in details and refer the reader to our paper [8]. Here we only present the results.

Let us remind first the form of the conventional quadratic in derivatives SYM action in 6 dimensions. It involves the $6D$ gauge field A_M , the gluino field ψ_i^a satisfying (3.3) and the triplet of auxiliary fields \mathcal{D}_{ik} . The action reads

$$S = \frac{1}{f^2} \int d^6x \operatorname{Tr} \left\{ -\frac{1}{2} F_{MN}^2 - \frac{1}{2} \mathcal{D}^{ik} \mathcal{D}_{ik} + i \psi^k \gamma_M \nabla_M \psi_k \right\}, \quad (3.8)$$

where f is the coupling constant of canonical dimension -1 and ∇_M is the covariant derivative.

If going down to four dimensions, one reproduces the action for $\mathcal{N} = 2$ 4D SYM theory. A_M gives the 4D gauge field A_μ and the adjoint scalar, ψ_i^a gives two 4D gluino fields while the triplet of auxiliary fields can be decomposed into the real auxiliary field D of the 4-dimensional $\mathcal{N} = 1$ vector multiplet and the complex auxiliary field F of the adjoint chiral multiplet.

The action of the HD 6D gauge theory was derived in [8]. The result is

$$\begin{aligned}
S = & -\frac{1}{g^2} \int d^6x \text{Tr} \left\{ (\nabla^M F_{ML})^2 + i\psi^j \gamma^M \nabla_M (\nabla^2 \psi_j) + \frac{1}{2} (\nabla_M \mathcal{D}_{jk})^2 \right. \\
& + \mathcal{D}_{lk} \mathcal{D}^{kj} \mathcal{D}_j{}^l - 2i\mathcal{D}_{jk} (\psi^j \gamma^M \nabla_M \psi^k - \nabla_M \psi^j \gamma^M \psi^k) + (\psi^j \gamma_M \psi_j)^2 \\
& \left. + \frac{1}{2} \nabla_M \psi^i \gamma^M \sigma^{NS} [F_{NS}, \psi_j] - 2\nabla^M F_{MN} \psi^j \gamma^N \psi_j \right\}. \quad (3.9)
\end{aligned}$$

The lagrangian has the canonical dimension 6 and the coupling constant g is dimensionless.

Let us discuss this result. Note first of all that the quadratic terms in the lagrangian are obtained from (3.8) by adding the extra box operator (it enters with negative sign, this makes the kinetic terms positive definite in Minkowski space). It is immediately seen for the terms $\propto \mathcal{D}^2$ and for the fermions. This is true also for the gauge part due to the identity

$$\text{Tr} \{ (\nabla^M F_{MN})^2 \} = -\frac{1}{2} \text{Tr} \{ F^{MN} \nabla^2 F_{MN} \} - 2i \text{Tr} \{ F_M{}^N F_{NS} F^{SM} \}. \quad (3.10)$$

The former auxiliary fields \mathcal{D}^{ik} become dynamical. They carry canonical dimension 2 and their kinetic term involves two derivatives. There is a cubic term $\propto \mathcal{D}^3$. This sector of the theory reminds the renormalizable theory $(\phi^3)_6$. Gauge and fermion fields have the habitual canonical dimensions $[A_M] = 1$, $[\psi] = 3/2$. Their kinetic terms involve, correspondingly, 4 and 3 derivatives. The lagrangian involves also other interaction terms, all of them having the canonical dimension 6.

It is instructive to evaluate the number of on-shell degrees of freedom for this lagrangian. Consider first the gauge field. With the standard lagrangian $\propto \text{Tr} \{ F_{MN}^2 \}$, a six-dimensional gauge field A_M has 4 on-shell d.o.f. for each color index. The simplest way to see this is to note that A_0 is not dynamical and we have to impose the Gauss law constraint on the remaining 5 spatial variables. For the higher-derivative theory, however, the presence of two extra derivatives doubles the number of d.o.f. and the correct counting is $2 \times 5 = 10$ before imposing the Gauss law constraint and $10 - 1 = 9$ after that. In addition, there are 3 d.o.f. of the fields D_{ij} and we have all together 12 bosonic d.o.f. for each color index. The standard 6D Weyl fermion (with the lagrangian involving only one derivative) has 4 on-shell degrees of freedom. In our case, we have $4 \times 3 = 12$ fermionic d.o.f. due to the presence of three derivatives in the kinetic term. Not unexpectedly, the numbers of bosonic and fermionic degrees of freedom on mass shell coincide.

3.1 Renormalization

The lagrangian (3.9) does not involve dimensional parameters and is scale-invariant. A less trivial and rather remarkable fact is that the action is also invariant with respect to special conformal transformations and the full superconformal group. This is true at the classical level, but, unfortunately, conformal invariance of this theory is broken by quantum effects. To see this, let us calculate (at the one-loop level) the β function of our theory.

The simplest way to do this calculation is to evaluate 1-loop corrections to the structures $\sim (\partial_M \mathcal{D})^2$ and $\sim \mathcal{D}^3$. The relevant Feynman graphs are depicted in Figs. 1, 2.

For perturbative calculations, we absorb the factor $1/g$ in the definition of the fields. The relevant propagators are

$$\begin{aligned}\langle A_M^A A_N^B \rangle &= -\frac{i\eta_{MN}\delta^{AB}}{p^4}, \\ \langle \psi^{jA} \psi^{kB} \rangle &= -\frac{i\epsilon^{jk}\delta^{AB} p_N \tilde{\gamma}^N}{p^4}, \\ \langle \mathcal{D}_{ik}^A \mathcal{D}_{jl}^B \rangle &= -\frac{i\delta^{AB}}{p^2} (\epsilon_{ij}\epsilon_{kl} + \epsilon_{il}\epsilon_{kj}),\end{aligned}\tag{3.11}$$

where A, B are color indices, $A_M = A_M^A t^A$, etc. The vertices can be read out directly from the lagrangian.

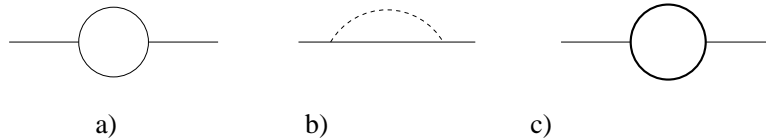


Figure 1: Graphs contributing to the renormalization of the kinetic term. Thin solid lines stand for the particle \mathcal{D} , thick solid lines for fermions, and dashed lines for gauge bosons.

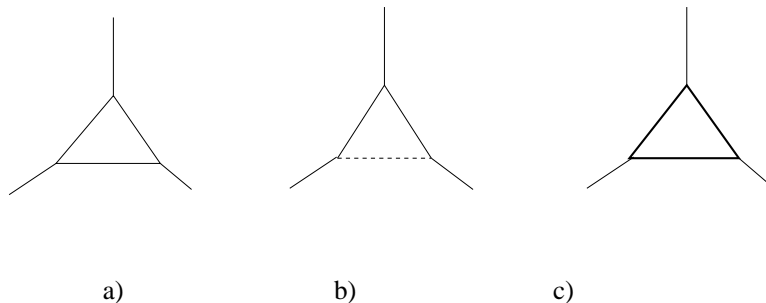


Figure 2: The same for the \mathcal{D}^3 vertex.

Consider first the graphs in Fig. 1. They involve logarithmic and quadratic divergences. The individual quadratically divergent contributions in the Wilsonian effective

lagrangian are

$$\begin{aligned}\Delta\mathcal{L}_{1a}^{\text{eff}} &= -\frac{9c_V}{2}\text{Tr}\{\mathcal{D}_{jk}^2\}I, \\ \Delta\mathcal{L}_{1b}^{\text{eff}} &= \frac{c_V}{2}\text{Tr}\{\mathcal{D}_{jk}^2\}I, \\ \Delta\mathcal{L}_{1c}^{\text{eff}} &= 4c_V\text{Tr}\{\mathcal{D}_{jk}^2\}I,\end{aligned}\tag{3.12}$$

where c_V is the adjoint Casimir eigenvalue and

$$I = \int^\Lambda \frac{d^6 p_E}{(2\pi)^6 p_E^4}.\tag{3.13}$$

We see that the quadratic divergences cancel out in the sum of the three graphs. The logarithmic divergences in the 2-point graphs are

$$\Delta\mathcal{L}_{(2)}^{\text{eff}} = g^2 c_V \left(-\frac{3}{2} - \frac{7}{6} + 2\right) \text{Tr}\{(\partial_M \mathcal{D}_{jk})^2\} L = -\frac{2g^2 c_V}{3} \text{Tr}\{(\partial_M \mathcal{D}_{jk})^2\} L,\tag{3.14}$$

where

$$L = \int_\mu^\Lambda \frac{d^6 p_E}{(2\pi)^6 p_E^6} = \frac{1}{64\pi^3} \ln \frac{\Lambda}{\mu}\tag{3.15}$$

and three terms in the parentheses correspond to the contributions of the graphs in Fig. 1a,b,c.

The 3-point graphs in Fig. 2 involve only logarithmic divergence. We obtain

$$\Delta\mathcal{L}_{(3)}^{\text{eff}} = g^3 c_V \left(-\frac{9}{2} - \frac{3}{2} + \frac{32}{3}\right) \text{Tr}\{\mathcal{D}_{lk} \mathcal{D}^{kj} \mathcal{D}_j^l\} L = \frac{14g^3 c_V}{3} \text{Tr}\{\mathcal{D}_{lk} \mathcal{D}^{kj} \mathcal{D}_j^l\} L.\tag{3.16}$$

The full 1-loop effective lagrangian in the \mathcal{D} sector is

$$\mathcal{L}_{\mathcal{D}}^{\text{eff}} = -\frac{1}{2} \text{Tr}\{(\partial_M \mathcal{D}_{jk})^2\} \left(1 + \frac{4g^2 c_V}{3} L\right) - g \text{Tr}\{\mathcal{D}_{lk} \mathcal{D}^{kj} \mathcal{D}_j^l\} \left(1 - \frac{14g^2 c_V}{3} L\right).\tag{3.17}$$

Absorbing the renormalization factor of the kinetic term in the field redefinition, we finally obtain

$$g(\mu) = g_0 \left(1 - \frac{20g_0^2 c_V}{3} L\right) = g_0 \left(1 - \frac{5g_0^2 c_V}{48\pi^3} \ln \frac{\Lambda}{\mu}\right)\tag{3.18}$$

for the effective charge renormalization.

The sign corresponds to the Landau zero situation, as in the conventional QED.

4 Discussion

Our study was motivated by the dream or rather by a sequence of dreams spelled out in the Introduction. By the reasons outlined there

1. We *believe* that the TOE is a conventional field theory in multidimensional bulk.
2. We *believe* that our Universe represents a thin soap bubble — a classical 3-brane solution in this theory.
3. If the theory claims to be truly fundamental, it should be renormalizable. For $D > 4$, this means the presence of higher derivatives in the action.
4. We *believe* that for superconformal theories, a way to tackle the HD ghost trouble exists.
5. We *believe* (but not so firmly, this is just the most attractive possibility) that the TOE enjoys the maximum $\mathcal{N} = 2$ superconformal symmetry in six dimensions.

Besides dreams, there are also some positive results. First, we constructed a QM HD model where the problem of ghosts *is* resolved. Second, we constructed a nontrivial example of renormalizable higher-dimensional supersymmetric gauge theory. It is $6D, \mathcal{N}=(1, 0)$ gauge theory with four derivatives in the action and dimensionless coupling constant.

Our theory enjoys superconformal invariance at the classical level, but, unfortunately, the superconformal symmetry is anomalous in this case. As the result of this breaking, in accord with the arguments of [2], the quantum theory suffers from ghosts which can hardly be harmless.

Four-dimensional experience teaches us that though nonsupersymmetric, $\mathcal{N} = 1$, and $\mathcal{N} = 2$ supersymmetric theories are anomalous, the maximum $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is truly conformal — β function vanishes there. It is very natural therefore to *believe* that unconstructed yet Holy Grail $\mathcal{N} = (2, 0)$ maximum superconformal $6D$ theory is free from anomaly.

How can it look like ? The first idea coming to mind is to ape the $4D$ construction and to couple the $6D$ gauge supermultiplet to $6D$ hypermultiplets. Adding this term to (3.9) one might hope to obtain a theory which would enjoy extended superconformal symmetry. Unfortunately, this program meets serious technical difficulties and it is not clear at the moment whether it can be carried out.

The second possibility is that the $\mathcal{N} = (2, 0)$ theory does not involve at all the gauge supermultiplet with the action (3.9), but depends on tensor rather than vector multiplets [7, 16]. Unfortunately, to describe the tensor multiplet in the framework of HSS is not a trivial task which is not solved yet. As a result, no microscopic lagrangian for interacting $(2, 0)$ tensor multiplet is known today...

Finally, one cannot exclude a disappointing possibility that the $(2, 0)$ theory does not have a lagrangian formulation whatsoever.

But the hope dies last !

References

- [1] See e.g. S. Hawking, *Comm. Math. Phys.* **87**, 395 (1982); C.J. Isham, *Canonical Quantum Gravity and the Problem of Time*, gr-qc/9210011, published in the Proceedings of GIFT Int. Seminar on Theor. Physics, Salamanca, 15-27 June, 1992; A. Carlini and I.D. Novikov, *Int. J. Phys.* **D5**, 445 (1996) [gr-qc/9607063].
- [2] A.V. Smilga, *Nucl. Phys.* **B 706**, 598 (2005) [hep-th/0407231]
- [3] K. Akama, *Lect. Notes Phys.* **176**, 267 (1982); V.A. Rubakov and M.E. Shaposhnikov, *Phys. Lett.* **B 125**, 136 (1983).
- [4] A. Pais and G.E. Uhlenbeck, *Phys. Rev.* **79**, 145 (1950).
- [5] A.V. Smilga, hep-th/0503213, to be published in *Nucl. Phys. B*.
- [6] S. Minwalla, *Adv. Theor. Math. Phys.* **2**, 781 (1998) [hep-th/9712074]; A. Van Proeyen, “*Tools for supersymmetry*”, hep-th/9910030.
- [7] P. Claus, R. Kallosh and A. Van Proeyen, *Nucl. Phys.* **B 518**, 117 (1998) [hep-th/9711161]; O. Aharony, M. Berkooz and N. Seiberg, *Adv. Theor. Math. Phys.* **2**, 119 (1998) [hep-th/9712117]; F. Bastianelli, S. Frolov and A.A. Tseytlin, *JHEP* **0002**, 013 (2000) [hep-th/0001041]; G. Arutyunov and E. Sokatchev, *Nucl. Phys.* **B 635**, 3 (2002) [hep-th/0201145].
- [8] E.A. Ivanov, A.V. Smilga, and B.M. Zupnik, hep-th/0505082.
- [9] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Pis'ma ZhETF* **40**, 155 (1984) [*JETP Lett.* **40**, 912 (1984)]; A.S. Galperin, E.A. Ivanov, S. Kalitzin, V.I. Ogievetsky and E.S. Sokatchev, *Class. Quantum Grav.* **1**, 469 (1984); A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press, 2001, 306 p.
- [10] P.D. Mannheim and A. Davidson, *Phys. Rev.* **A71**:042110, 2005 [hep-th/0408104].
- [11] A. Peres, *Ann. Phys.* **129**, 33 (1980).
- [12] M. Ostrogradsky, *Mem. Ac. St. Petersburg* **VI 4**, 385 (1850).
- [13] T. Nakamura and S. Hamamoto, *Progr. Theor. Phys.* **95**, 469 (1996) [hep-th/9511219].
- [14] V. de Alfaro and T. Regge, *Potential Scattering*, Amsterdam (1965).
For a recent discussion, see A.E. Shabad, hep-th/0403177.
- [15] P.S. Howe, G. Sierra and P.K. Townsend, *Nucl. Phys.* **B 221**, 331 (1983).

- [16] E. Witten, “*Some comments on string dynamics*”, In *Los Angeles 1995, Future perspectives in String Theory* 501-523, hep-th/9507121; A. Strominger, Phys. Lett. **B 383**, 44 (1996) [hep-th/9512059]; N. Seiberg, Nucl. Phys. Proc. Suppl. **67**, 158 (1998) [hep-th/9705117].